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DERIVATION PROPERTIES IN PRIME NEAR-RINGS

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Abstract

In this paper, N will denote a zero-symmetric left near-ring. A near-ring N is called a prime near-ring, if N has the property that for $a, b \in N$, $aNb = \{0\}$ implies $a = 0$ or $b = 0$. N is called a semiprime near-ring, if N has the property that for $a \in N$, $aNa = \{0\}$ implies $a = 0$. We will investigate some properties of derivations in prime and semiprime near-rings as Theorems 2.5, 2.7, and 2.9.

1. Introduction

Throughout this paper, N will denote a zero-symmetric left near-ring. A near-ring N is called a *prime near-ring*, if N has the property that for $a, b \in N$, $aNb = \{0\}$ implies $a = 0$ or $b = 0$. N is called a *semiprime near-ring*, if N has the property that for $a \in N$, $aNa = \{0\}$ implies $a = 0$.

A nonempty subset U of N is called a *right N-subset* (resp., *left N-subset*), if $UN \subset U$ (resp., $NU \subset U$), and if U is both a right N-subset

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and a left N-subset, it is said to be an *N-subset* of N. Every right ideal and right semigroup ideal of N are right N-subsets of N, and symmetrically, we can apply for left case. A *derivation* D on N is an additive endomorphism of N with the property that for all $a, b \in N$, $D(ab) = aD(b) + D(a)b$.

All other basic properties, terminologies, and concepts are can be found in the book of Pilz [3].

2. Properties of Derivations in Near-Rings

Lemma 2.1. *Let D be an arbitrary additive endomorphism of N. Then $D(ab) = aD(b) + D(a)b$ for all $a, b \in N$, if and only if $D(ab) = D(a)b + aD(b)$ for all $a, b \in N$.*

Proof. Suppose that $D(ab) = aD(b) + D(a)b$, for all $a, b \in N$.

From $a(b + b) = ab + ab$, and since N satisfies left distributive law

$$\begin{aligned} D(a(b+b)) &= aD(b+b) + D(a)(b+b) \\ &= a(D(b) + D(b)) + D(a)b + D(a)b \\ &= aD(b) + aD(b) + D(a)b + D(a)b, \text{ and} \end{aligned}$$

$$D(ab + ab) = D(ab) + D(ab) = aD(b) + D(a)b + aD(b) + D(a)b.$$

Comparing these two equalities, we have $aD(b) + D(a)b = D(a)b + aD(b)$. Hence, $D(ab) = D(a)b + aD(b)$.

Conversely, suppose that $D(ab) = D(a)b + aD(b)$, for all $a, b \in N$. Then from $D(a(b+b)) = D(ab + ab)$ and the above calculation of this equality, we can induce that $D(ab) = aD(b) + D(a)b$.

Lemma 2.2 [1]**.** *Let D be a derivation on N. Then N satisfies the following right distributive law: for all a, b, c in N*

$$\{aD(b) + D(a)b\}c = aD(b)c + D(a)bc,$$

$$\{D(a)b + aD(b)\}c = D(a)bc + aD(b)c.$$

Proof. From the calculation for $D((ab)c) = D(a(bc))$ and Lemma 2.1, we can induce our result.

Lemma 2.3. *Let N be a prime near-ring and let U be a nonzero N-subset of N. If x is an element of N such that $Ux = \{0\}$ (or $xU = \{0\}$), then $x = 0$.*

Proof. Since $U \neq \{0\}$, there exist an element $u \in U$ such that $u \neq 0$.

Consider that $uNx \subset Ux = \{0\}$. Since $u \neq 0$ and N is a prime near-ring, we have that $x = 0$.

Corollary 2.4. *Let N be a semiprime near-ring and let U be a nonzero N-subset of N. If x is an element of $N(U)$ such that $Ux^2 = \{0\}$ (or $x^2U = \{0\}$), then $x = 0$. Here $N(U)$ is the normalizer of U.*

Theorem 2.5. *Let N be a prime near-ring and U be a nonzero N-subset of N. If D is a nonzero derivation on N. Then,*

(i) *If $a, b \in N$ and $aUb = \{0\}$, then $a = 0$ or $b = 0$.*

(ii) *If $a \in N$ and $D(U)a = \{0\}$, then $a = 0$.*

(iii) *If $a \in N$ and $aD(U) = \{0\}$, then $a = 0$.*

Proof. (i) Let $a, b \in N$ and $aUb = \{0\}$. Then $aUNb \subset aUb = \{0\}$. Since N is a prime near-ring, $aU = 0$ or $b = 0$.

If $b = 0$, then we are done. So if $b \neq 0$, then $aU = 0$. Applying Lemma 2.3, $a = 0$.

(ii) Suppose $D(U)a = \{0\}$, for $a \in N$. Then, for all $u \in U$ and $b \in N$, from Lemma 2.2, we have

$0 = D(bu)a = (bD(u) + D(b)u)a = bD(u)a + D(b)ua = D(b)ua$. Hence, $D(b)Ua = \{0\}$ for all $b \in N$.

Since D is a nonzero derivation on N, we have that $a = 0$ by the statement (i).

(iii) Suppose $aD(U) = \{0\}$ for $a \in N$. Then, for all $u \in U$ and $b \in N$, $0 = aD(ub) = a\{uD(b) + D(u)b\} = auD(b) + aD(u)b = auD(b)$. Hence $aUD(b) = \{0\}$ for all $b \in N$.

From the statement (i) and since D is a nonzero derivation on N, we have that $a = 0$.

We remark that to obtain any of the conclusions of Theorem 2.5, it is not sufficient to assume that U is a right N-subset, even in the case that N is a ring. Consider the following example:

Example 2.6. Let R be the prime ring $Mat_2(F)$, where F is an arbitrary field. Let $U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R$ and let D be the inner derivation of R given by

$$D(w) = w\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} w.$$

Then, $$D(U) = \left\{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} R \mid a \in F\right\},$$

so that for $$x = y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

we have

$$xUy = xD(u) = D(U)x = \{0\}.$$

Theorem 2.7. *Let N be a prime near-ring and U be a right N-subset of N. If D is a nonzero derivation on N such that $D^2(U) = 0$, then $D^2 = 0$.*

Proof. For all $u, v \in U$, we have $D^2(uv) = 0$. Then,

$$0 = D^2(uv) = D(D(uv)) = D\{D(u)D(v) + uD(v)\} = D^2(u)v + D(u)D(v)$$
$$+ D(u)D(v) + uD^2(v) = D^2(u)v + 2D(u)D(v) + uD^2(v).$$

Thus, $2D(u)D(U) = \{0\}$ for all $u \in U$. From Lemma 2.5 (iii), we have $2D(U) = \{0\}$.

Now, for all $b \in N$ and $u \in U$, $D^2(ub) = uD^2(b) + 2D(u)D(b) + D^2(u)b$. Hence, $UD^2(b) = \{0\}$ for all $b \in N$. By Lemma 2.3, we have $D^2(b) = 0$ for all $b \in N$. Consequently, $D^2 = 0$.

Lemma 2.8. *Let D be a derivation of a prime near-ring N and a be an element of N.*

If $aD(x) = 0$ *for all* $x \in N$, *then either* $a = 0$ *or D is zero.*

Proof. Suppose that $aD(x) = 0$ for all $x \in N$. Replacing x by xy, we have that $aD(xy) = 0 = aD(x)y + axD(y)$ by Lemma 2.2. Then, $axD(y) = 0$ for all $x, y \in N$.

If D is not zero, that is, if $D(y) \neq 0$ for some $y \in N$, then, since N is a prime near-ring, $aND(y)$ implies that $a = 0$.

Now, we prove our main result, which extends a famous theorem of Posner on rings [4] to near-rings with some condition.

Theorem 2.9. *Let N be a prime near-ring of* 2*-torsion free and let* D_1 *and* D_2 *be derivations on N such that* D_1D_2 *is also a derivation on N with the condition* $D_1(a)D_2(b) = D_2(b)D_1(a)$, *for all* $a, b \in N$. *Then either* $D_1 = 0$ *or* $D_2 = 0$.

Proof. Since D_1D_2 is a derivation, we have

$$D_1D_2(ab) = aD_1D_2(b) + D_1D_2(a)b. \tag{1}$$

Also, since D_1 and D_2 are derivations, we get

$$\begin{aligned} D_1D_2(ab) &= D_1(D_2(ab)) = D_1(aD_2(b) + D_2(a)b) = D_1(aD_2(b)) \\ &\quad + D_1(D_2(a)b) = aD_1D_2(b) + D_1(a)D_2(b) \\ &\quad + D_2(a)D_1(b) + D_1D_2(a)b. \end{aligned} \tag{2}$$

From (1) and (2), for $D_1D_2(ab)$ for all $a, b \in N$,

$$D_1(a)D_2(b) + D_2(a)D_1(b) = 0. \tag{3}$$

Replacing a by $aD_2(c)$ in (3), and using Lemmas 2.1 and 2.2, we obtain that

$$\begin{aligned} 0 &= D_1(aD_2(c))D_2(b) + D_2(aD_2(c))D_1(b) \\ &= \{D_1(a)D_2(c) + aD_1D_2(c)\}D_2(b) + \{aD_2^2(c) + D_2(a)D_2(c)\}D_1(b) \\ &= D_1(a)D_2(c)D_2(b) + aD_1D_2(c)D_2(b) + aD_2^2(c)D_1(b) + D_2(a)D_2(c)D_1(b) \\ &= D_1(a)D_2(c)D_2(b) + a\{D_1D_2(c)D_2(b) + D_2^2(c)D_1(b)\} + D_2(a)D_2(c)D_1(b). \end{aligned}$$

In this last equality,

$$a\{D_1D_2(c)D_2(b) + D_2^2(c)D_1(b)\} = 0.$$

We obtain that $D_1D_2(c)D_2(b) + D_2^2(c)D_1(b) = 0$ by replacing a by $D_2(c)$ in (3). Hence, we have the following equality: for all $a, b, c \in N$,

$$D_1(a)D_2(c)D_2(b) + D_2(a)D_2(c)D_1(b) = 0. \tag{4}$$

Replacing a and b by c in (3), respectively, we see that

$$D_2(c)D_1(b) = -D_1(c)D_2(b),$$

$$D_1(a)D_2(c) = -D_2(a)D_1(c).$$

Now, (4) becomes

$$\begin{aligned} 0 &= \{-D_2(a)D_1(c)\}D_2(b) + D_2(a)\{-D_1(c)D_2(b)\} \\ &= D_2(a)(-D_1(c))D_2(b) + D_2(a)(-D_1(c))D_2(b) \\ &= D_2(a)\{(-D_1(c))D_2(b) - D_1(c)D_2(b)\}, \end{aligned}$$

for all $a, b, c \in N$.

If $D_2 \neq 0$, then by Lemma 2.8, we have the equality:

$(-D_1(c))D_2(b) - D_1(c)D_2(b) = 0$, that is,

$$D_1(c)D_2(b) = (-D_1(c))D_2(b), \text{ for all } b, c \in N. \tag{5}$$

On the other hand, using the given condition of our theorem,

$$(-D_1(c))D_2(b) = D_1(-c)D_2(b) = D_2(b)D_1(-c)$$

$$= D_2(b)(-D_1(c)) = -D_2(b)D_1(c) = -D_1(c)D_2(b). \tag{6}$$

From (5) and (6), we have that for all $b, c \in N$, $2D_1(c)D_2(b) = 0$. Since N is of 2-torsion free, $D_1(c)D_2(b) = 0$. Also, since D_2 is not zero, by Lemma 2.8, we see that $D_1(c) = 0$ for all $c \in N$. Therefore, $D_1 = 0$. Consequently, either $D_1 = 0$ or $D_2 = 0$. Thus our proof is complete.

As a consequence of our Theorem 2.9, we get the following important statement:

Corollary 2.10. *Let N be a prime near-ring of* 2*-torsion free, and let D be a derivation on N such that* $D^2 = 0$. *Then* $D = 0$.

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